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Brane charges and Chern–Simons invariants of hyperbolic spaces, with cosmological applications*

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Abstract
We discuss methods of K-theory associated with hyperbolic orbifolds and valid for the description of Chern morphisms and brane charges. Such methods of K-theory are applied to compute D-brane charges, which are identified with elements of Grothendick K-groups, and for manifolds with horizons, spaces that naturally arise as the near-horizon of black brane geometries. In de Sitter spaces, these solutions break supersymmetry, and do not describe universes with zero cosmological constant. Here we pay attention to real hyperbolic spaces, and we examine associated Chern classes and brane charges using methods of K-theory and spectral theory of differential operators related to real hyperbolic spaces. An argument in favour of hyperbolic geometries in the treatment of the contributions to the vacuum persistence amplitude in QFT is given. All those are to be viewed as the proper mathematical structures underlying QFT with relevant backgrounds and boundary conditions in string cosmology.

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1. Introduction

In superstring theories, D-branes play a significant role. Methods of K-theory have been applied to compute D-brane charges [1–3] which are identified with elements of Grothendick K-groups [4–6]. The relevant description of the Ramond–Ramond charges in terms of

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equivariant K-theory has been demonstrated in [1, 7–9]. It leads to a formalism of fractional branes pinned on the orbifold singularities which have components in the twisted sectors of the closed strings. In string compactifications, de Sitter, anti de Sitter spaces and N-spheres play an important role. These spaces also naturally arise as the near-horizon of black brane geometries. Spheres and anti de Sitter spaces, as supergravity solutions, have been extensively studied. But not many investigations have been carried out for de Sitter and hyperbolic spaces. As for de Sitter spaces, the reasons are that these solutions break supersymmetry, and do not describe universes with zero cosmological constant. In this paper, we pay attention to real hyperbolic spaces. For the K-theory interpretation of brane charges, the Chern isomorphism can be used. The rational cohomology ring $H^{\mathrm{even}}(X, \mathbb{Q})$ over a manifold, X, has a natural inner product, while the pairing $K(X)$, associated with the cohomology ring $K(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, is given by the index of the Dirac operator. We examine the Chern classes and charges of branes using methods of K-theory and the spectral theory of differential operators related to real hyperbolic spaces.

2. $U(n)$-Chern–Simons invariants

We consider real hyperbolic spaces which can arise as horizons in string compactifications [10]. Let $X = G/\mathcal{K}$ be an irreducible rank 1 symmetric space of non-compact type. G will thus be a connected non-compact simple split rank 1 Lie group with finite centre, and $\mathcal{K} \subset G$ is a maximal compact subgroup. The objects of interest are the groups $G = SO_1(n, 1)(n \in \mathbb{Z}_+)$ and $\mathcal{K} = SO(n)$. The corresponding symmetric space of non-compact type is the real hyperbolic space $X = \mathbb{H}^n = SO_1(n, 1)/SO(n)$ of sectional curvature -1. Let $X_\Gamma = \Gamma\backslash G/\mathcal{K}$ be a real compact hyperbolic manifold. The fundamental group of X_Γ acts by covering transformations on X and gives rise to a discrete, co-compact subgroup $\Gamma \subset G$. Suppose that χ is a one-dimensional representation of Γ factors through a representation of $H^1(X; \mathbb{Z})$. It can be shown that, for a unitary representation $\chi : \Gamma \to U(n)$, the corresponding flat vector bundle $\mathbb{E}_\chi$ is topologically trivial ($\mathbb{E}_\chi \cong X \otimes \mathbb{C}^n$) if and only if $\det\chi|_{\mathrm{Tor}^1} : \mathrm{Tor}^1 \to U(1)$ is the trivial representation. Here Tor^1 is the torsion part of $H^1(M; \mathbb{Z})$ and $\det \chi$ is a one-dimensional representation of Γ, defined by $\det \chi(\gamma) := \det(\chi(\gamma))$ for $\gamma \in \Gamma$.

A 3-form flux associates a phase with a Euclidean brane world-volume, X_Γ, which is given by the eta invariant, $\eta(0)$, of the virtual bundle restricted to X_Γ. We can express this phase directly in terms of the Chern–Simons invariant. One can construct a vector bundle $\widetilde{\mathbb{E}}_\chi$ over a certain 4-manifold, M, which is an extension of a flat complex vector bundle $\mathbb{E}_\chi$ over X_Γ. The relevant cobordism group vanishes and, in fact, for a 3-manifold X_Γ we can find a 4-manifold M and extend the $U(n)$ bundle such that the manifold M is spin. For any extension $\widetilde{A}_\chi$ of a flat connection A_χ corresponding to χ, the second Chern character $\mathrm{ch}_2(\widetilde{\mathbb{E}}_\chi) \big(= -(1/8\pi^2)\,\mathrm{Tr}\big(F_{\widetilde{A}_\chi} \wedge F_{\widetilde{A}_\chi}\big)\big)$ of $\widetilde{\mathbb{E}}_\chi$ can be expressed in terms of the first and second Chern classes: $\mathrm{ch}_2(\widetilde{\mathbb{E}}_\chi) = \frac{1}{2}c_1(\widetilde{\mathbb{E}}_\chi)^2 - c_2(\widetilde{\mathbb{E}}_\chi)$. The Chern character and the $\widehat{A}$-genus, the usual polynomial related to the Riemannian curvature, Ω, are given by

$$\begin{aligned} &\mathrm{ch}(\widetilde{\mathbb{E}}_\chi) = \mathrm{rank}\,\widetilde{\mathbb{E}}_\chi + c_1(\widetilde{\mathbb{E}}_\chi) + \mathrm{ch}_2(\widetilde{\mathbb{E}}_\chi) = \dim\chi + c_1(\widetilde{\mathbb{E}}_\chi) + \mathrm{ch}_2(\widetilde{\mathbb{E}}_\chi), \\ &\widehat{A}(\Omega^M) = 1 - \tfrac{1}{24}p_1(\Omega^M). \end{aligned} \tag{1}$$

Here, $p_1(\Omega^M)$ is the first Pontryagin class and Ω^M is the Riemannian curvature of the 4-manifold M, which has a boundary $\partial M = X_\Gamma$. Thus, we have

$$\begin{aligned} \mathrm{ch}(\widetilde{\mathbb{E}}_\chi)\widehat{A}(\Omega^M) &= (\dim\chi + c_1(\widetilde{\mathbb{E}}_\chi) + \mathrm{ch}_2(\widetilde{\mathbb{E}}_\chi))\left(1 - \frac{1}{24}p_1(\Omega^M)\right) \\ &= \dim\chi + c_1(\widetilde{\mathbb{E}}_\chi) + \mathrm{ch}_2(\widetilde{\mathbb{E}}_\chi) - \frac{\dim\chi}{24}p_1(\Omega^M). \end{aligned} \tag{2}$$

The integral over the manifold M takes the form

$$\int_M \mathrm{ch}(\widetilde{\mathbb{E}}_\chi)\widehat{A}(\Omega^M) = \int_M \mathrm{ch}_2(\widetilde{\mathbb{E}}_\chi) - \frac{\dim\chi}{24}\int_M p_1(\Omega^M). \tag{3}$$

The intersection form on M allows one to define the Chern–Simons invariant using c_1^2. For the Chern classes, we have $c_1(\mathbb{E}_\chi) \in H^2(X_\Gamma, \mathbb{Z})$, $c_2(\mathbb{E}_\chi) \in H^4(X_\Gamma, \mathbb{Z})$ and $H^4(X_\Gamma, \mathbb{Z}) \cong \mathbb{Z}_{|\Gamma|}$. The Dirac index is given by [11]

$$\mathrm{Index}\, D_{\widetilde{A}_\chi} = \int_M \mathrm{ch}(\widetilde{\mathbb{E}}_\chi)\widehat{A}(M) - \frac{1}{2}(\eta(0,\mathfrak{D}_\chi) + h(0,\mathfrak{D}_\chi)), \tag{4}$$

where $h(0,\mathfrak{D}_\chi)$ is the dimension of the space of harmonic spinors on X_Γ ($h(0,\mathfrak{D}_\chi) = \dim\mathrm{Ker}\,\mathfrak{D}_\chi$ = multiplicity of the 0-eigenvalue of $\mathfrak{D}_\chi$ acting on X_Γ); $\mathfrak{D}_\chi$ is a Dirac operator on X_Γ acting on spinors with coefficients in χ. The Chern–Simons action, $CS_{U(n)}(\widetilde{A}_\chi) = -(1/8\pi^2)\int_M \mathrm{Tr}\big(F_{\widetilde{A}_\chi}\wedge F_{\widetilde{A}_\chi}\big)$, can be derived from equation (4). Indeed,

$$\mathrm{Index}\, D_{\widetilde{A}_\chi} = CS(\widetilde{A}_\chi) - \frac{\dim\chi}{24}\int_M p_1(\Omega^M) - \frac{1}{2}[\eta(0,\mathfrak{D}_\chi) + h(0,\mathfrak{D}_\chi)]. \tag{5}$$

There exists a Selberg-type (Shintani) zeta function $Z(s,\mathfrak{D}_\chi)$ associated with a twisted Dirac operator $\mathfrak{D}_\chi$ acting on oriented odd-dimensional real hyperbolic spaces. $Z(s,\mathfrak{D}_\chi)$ is a meromorphic function of $s\in\mathbb{C}$, and for $\Re(s^2)\gg 0$ one has [12] $\log Z(0,\mathfrak{D}_\chi) = \sqrt{-1}\pi\eta(0,\mathfrak{D}_\chi)$. Thus, finally we get the $U(n)$-Chern–Simons invariant of an irreducible flat connection on the real hyperbolic 3-manifolds:

$$\begin{aligned} CS(\widetilde{A}_\chi) - \mathrm{modulo}(\mathbb{Z}/2) &= \frac{1}{2}[\dim\chi\,\eta(0,\mathfrak{D}) - \eta(0,\mathfrak{D}_\chi)] \\ &= \frac{1}{2\pi\sqrt{-1}}\log\left[\frac{Z(0,\mathfrak{D})^{\dim\chi}}{Z(0,\mathfrak{D}_\chi)}\right]. \end{aligned} \tag{6}$$

The value of the Chern–Simons functional on the space of connections at a critical point can be regarded as a topological invariant of a pair (X_Γ, χ). Note that using K-theory the groups of fluxes can be computed, confirming our computation via Chern–Simons invariants (see [13]). Indeed if the one-form flux is zero and $c_1(\mathbb{E}_\chi) = 0$ then the flux is measured by $c_2(\mathbb{E}_\chi) \in H^3(X_\Gamma, U(1))$. This group of three-form fluxes is generated by the two-dimensional representation and is given by [13] $H^4(X_\Gamma,\mathbb{Z}) = H^3(X_\Gamma, U(1)) = \mathbb{Z}_{|\Gamma|}$.

3. Brane charges

Before discussing the brane charge formula, we begin with some conventions which apply throughout. Let X be an oriented manifold, and let $H^*(X)$ be the cohomology ring of X. The Poincaré duality (a well-known result in differential topology) gives a canonical isomorphism

$$\mathfrak{d}_X : H^j(X) \xrightarrow{\approx} H_{j-p}(X), \qquad \text{for all} \quad p = 0, 1, \ldots, n = \dim X. \tag{7}$$

Let $f : Y \to X$ be a continuous map from Y to X and $m = \dim Y$. For all $p \geqslant m-n$ there is a linear map, called the Gysin homomorphism: $f_! : H^j(Y) \longrightarrow H^{j-(m-n)}(X)$, which is defined such that the sequence

$$\begin{aligned} &H^p(Y) \xrightarrow{\mathfrak{d}_Y} H_{m-p}(Y) \xrightarrow{f_*} H_{m-p}(X) \\ &H_{m-p}(X) \xrightarrow{\mathfrak{d}_X^{-1}} H^{p-(m-n)}(X) H^{p-(m-n)}(X) \xleftarrow{f_!} H^p(Y) \end{aligned} \tag{8}$$

is commutative. Thus, $f_! = \mathfrak{d}_X^{-1} f_* \mathfrak{d}_Y$, where f_* is the natural push-forward map acting on homology. As an example of that construction, let us assume that Y is an oriented vector bundle E over X, of fibre dimension ℓ. The canonical projection map, $\pi : E \to X$, and the inclusion $\iota : X \to E$ of the zero section induce maps on the homology with $\pi_* \iota_* = \mathrm{Id}$. For all j, we have the following isomorphisms: $\pi_! : H^{j+\ell}(E) \xrightarrow{\approx} H^j(X)$, $\iota_! : H^j(X) \xrightarrow{\approx} H^{j+\ell}(E)$. $\pi_!$ is the Gysin map; it can be associated with integration over the fibres of $E \to X$. We have $\pi_! \iota_! = \mathrm{Id}$, so that $\pi_! = (\iota_!)^{-1}$. The map $i_!$ is called the Thom isomorphism of the oriented vector bundle E. The particular example $j = 0$ is an important case of the Thom isomorphism. For $j = 0$, a map $H^0(X) \to H^\ell(E)$ and the image of $1 \in H^0(X)$ determine a cohomology class $\iota_!(1) \in H^\ell(E)$, which is called the Thom class of E.

Let us consider $U(n)$, a gauge bundle $\mathbb{E}$ on the brane. It has been shown that, as an element of $H^*(X)$, the Ramond–Ramond charge associated with a D-brane wrapping a supersymmetric cycle in spacetime $f : Y \hookrightarrow X$ with the Chan–Paton bundle $\mathbb{E} \to Y$, is given by

$$Q = \mathrm{ch}(f_! \mathbb{E}) \wedge [\widehat{A}(TX)]^{1/2}. \tag{9}$$

The map

$$\mathrm{ch} : K(X) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow H^{\mathrm{even}}(X, \mathbb{Q}) \equiv \bigoplus_{n \geqslant 0} H^{2n}(X, \mathbb{Q}) \tag{10}$$

is an isomorphism, and it can be extended to a ring isomorphism [15], $\mathrm{ch} : K^*(X) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\approx} H^*(X, \mathbb{Q})$, which maps $K^{-1}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ onto $H^{\mathrm{odd}}(X, \mathbb{Q})$. One of the features of the topological K-theory which makes it so useful in a variety of applications is the existence of the Chern character isomorphism. It is also one of the key properties of cyclic cohomology. Note that the rational cohomology ring $H^{\mathrm{even}}(X, \mathbb{Q})$ has a natural inner product, while the pairing $K(X)$, associated with the cohomology ring $K(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, is given by the index of the Dirac operator.

The result (9) is in complete agreement with the fact that the D-brane charge is given by $f_![\mathbb{E}] \in K(X)$, and it gives an explicit formula for the brane charges in terms of the Chern character isomorphism on K-theory. The Chern characters ch^* (cohomology) and ch_* (homology) preserve the 'cap' product $\cap$. It means that for every *topological space X* there is a $\mathbb{Z}_2$-degree preserving commutative sequence [16, 17]:

$$\begin{array}{l} K^*(X) \otimes K_*(X) \xrightarrow{\cap} K_*(X) \xrightarrow{\mathrm{ch}_*} H_*(X, \mathbb{Q}) \\ H_*(X, \mathbb{Q}) \xleftarrow{\cap} H^*(X, \mathbb{Q}) \otimes H_*(X, \mathbb{Q}) \xleftarrow{\mathrm{ch}^* \otimes \mathrm{ch}_*} K^*(X) \otimes K_*(X). \end{array} \tag{11}$$

For a finite CW-complex X, $K_*(X)$ is a finitely generated Abelian group and ch_* induces an isomorphism $K_*(X) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow H_*(X, \mathbb{Q})$ of $\mathbb{Z}_2$-graded vector spaces over $\mathbb{Q}$.

4. Methods of the algebraic *K*-theory of hyperbolic orbifolds

In this section, we discuss a computation of the K-groups of the twisted group C^*-algebras which are relevant to the branes on hyperbolic orbifold singularities. The precise definitions are somewhat technical (see, for example, [4, 18]) and thus some mathematical precision has been skipped in the following discussion.

The main object here is the group $KK^G(A, B)$, which depends on a pair of graded G-algebras, A and B. Let A and B be C^*-algebras (recall a C^*-algebra is a Banach algebra with an involution satisfying $\|a^*\| = \|a\|^2$). A pair $(\mathcal{E}, \pi)$, where $\mathcal{E}$ is a $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert B-module acted upon by A through a *-homomorphism $\pi : A \to \mathcal{L}(E) = \mathrm{End}^*(\mathcal{E}), \forall a \in A$, the operator $\pi(a)$ being of degree $0, \pi(A) \subset \mathcal{L}(\mathcal{E})^{(0)}$, will be called an (A, B)-bimodule. Let $E(A, B)$ be a triple $(\mathcal{E}, \pi, F)$, where $(\mathcal{E}, \pi)$ is an (A, B)-module, $F \in \mathcal{L}(\mathcal{E})$ is a homogeneous operator of degree 1, and $\forall a \in A$: (i) $\pi(a)(F^2 - 1) \in \mathcal{K}(\mathcal{E})$, and (ii) $[\pi(a), F] \in \mathcal{K}(\mathcal{E}) (\mathcal{K}(\mathcal{E})$ is the algebra of compact operators). A triple $(\mathcal{E}, \pi, F)$ will be called degenerate if $\forall a \in A : \pi(a)(F^2 - 1) = 0, [\pi(a), F] = 0$. Let $D(A, B)$ be a set of generated triples. An element $E(A, B[0, 1])$, where $B[0, 1]$ is an algebra of continuous functions in B on the interval $[0, 1]$, will be called an homotopy in $E(A, B)$. Let us assign a direct sum in $E(A, B) : (\mathcal{E}, \pi, F) \oplus (\mathcal{E}', \pi', F') = (\mathcal{E} \oplus \mathcal{E}', \pi \oplus \pi', F \oplus F')$.

$KK(A, B)$ is an Abelian group and its elements are homotopy classes of $E(A, B)$. A *-homomorphism, $f : A_1 \to A_2$, transfers (A_2, B)-modules into (A_1, B)-modules, and [19] $f^* : E(A_2, B) \to E(A_1, B), (\mathcal{E}, \pi, f) \mapsto (\mathcal{E}, \pi^\circ f, F)$, while a $(*)$-homomorphism $g : B_1 \to B_2$ induces a homomorphism $g_* : E(A, B_1) \to E(A, B_2), (\mathcal{E}, \pi, f) \mapsto (\mathcal{E} \otimes_g B_2, \pi \otimes 1, F \otimes 1)$, where $\pi \otimes 1 : A \to \mathcal{L}(\mathcal{E} \otimes_g B_2), (\pi \otimes 1)(a)(e \otimes b) = \pi(a)e \otimes b$. The groups $KK(A, B)$ define a homotopy invariant bifunctor from the category of separable C^*-algebras into the category of Abelian groups. Abelian groups $KK(A, B)$ depend covariantly on the algebras A and B, in addition $KK(\mathbb{C}, B) = K_0(B)$. Of interest to us are the following relations:

$$KK^*(A := \mathbb{C}, B) = K_*(B), \qquad KK^*(A, B := \mathbb{C}) = K^*(A). \tag{12}$$

Assuming that $KK_j(A, B) = KK(A, B(\mathbb{R}^j))$, one can determine the higher KK-groups.

Let $1_A \in KK(A, A)$ ($KK(A, A)$ is a ring with unit) denote the triple class $(A, \iota_A, 0)$, where $A^{(1)} = A, A^{(0)} = 0$ and $\iota_A : A \to \mathcal{K}(A) \subset \mathcal{L}(A), \iota_A(a)b = ab, a, b \in A$. Consider also the map $\tau_D : KK(A, B) \otimes KK(A \otimes D, B \otimes D), \tau_D(\text{class}(\mathcal{E}, \pi, F)) = \text{class}(\mathcal{E} \otimes D, \pi \otimes 1_D, F \otimes 1)$. Finally we can determine Kasparov's pairing

$$KK(A, D) \times KK(D, B) \longrightarrow KK(A, B), \tag{13}$$

which, denoting $(x, y) \mapsto x \otimes_D y$, satisfies the following conditions.

(i) The Kasparov pairing depends covariantly on the algebra B and contravariantly on the algebra A.

(ii) If $f : D \to E$ is a $*$-homomorphism, then $f_*(x) \otimes_E y = x \otimes_D f^*(y), x \in KK(A, D), y \in KK(E, B)$.

(iii) Associative property: $(z \otimes_D y) \otimes_E z = x \otimes_D (y\otimes)_E z, \forall x \in KK(A, D), y \in KK(D, E), z \in KK(E, B)$.

(iv) $x \otimes_B 1_B = 1_A \otimes x = x, \qquad \forall x \in KK(A, B)$.

(v) $\tau_E(x \otimes_B y) = \tau_E(x) \otimes_{B\otimes E} \tau_E(y), \qquad \forall x \in KK(A, B), \qquad \forall y \in KK(B, D)$.

Suppose that for two algebras, A and B, there are elements $\alpha \in KK(A \otimes B, \mathbb{C}), \beta \in KK(\mathbb{C}, A \otimes B)$, with the property that $\beta \otimes_A \alpha = 1_B \in KK(B, B), \beta \otimes_B \alpha = 1_A \in KK(A, A)$. Then, the KK-duality isomorphisms between the K-theory (K-homology) of algebra A and the K-homology (K-theory) of algebra B occur: $K_*(A) \cong K^*(B), K^*(A) \cong K_*(B)$. In fact the algebras A and B are Poincaré dual [18], but generally speaking these algebras are not KK-equivalent.

We now review the concept of K-amenable groups following [20]. Let G be a connected Lie group and $\mathcal{K}$ is a maximal compact subgroup. We also assume that dim $(G/\mathcal{K})$ is even and $G/\mathcal{K}$ admits a G-invariant Spin$^{\mathbb{C}}$ structure. The G-invariant Dirac operator $\mathfrak{D} := \gamma^\mu \partial_\mu$ on $C/\mathcal{K}$ is a first-order self-adjoint, elliptic differential operator acting on L^2 sections of the $\mathbb{Z}_2$-graded homogeneous bundle of spinors $\mathcal{S}$. Let us consider a zeroth-order pseudo-differential operator $\mathcal{O} = \mathfrak{D}(1 + \mathfrak{D}^2)^{-1}$ acting on $H = L^2(G/\mathcal{K}, \mathcal{S})$. Suppose that $C_0(G/\mathcal{K})$ acts on H by multiplication of operators. G acts on $C_0(G/\mathcal{K})$ and H by left translation, and $\mathcal{O}$ is G-invariant. Then, the set $(\mathcal{O}, H, X)$ defines a canonical Dirac element $\alpha_G = KK_G(C_0(G/\mathcal{K}), \mathbb{C})$. There is a canonical *Mishenko element* $\alpha_G \in KK_G(C_0(G/\mathcal{K}), \mathbb{C})$ such that the following intersection products occur:

(i) $\alpha_G \otimes_{\mathbb{C}} \beta_G = 1_{C_0(G/\mathcal{K})} \in KK_G(C_0(G/\mathcal{K}), C_0(G/\mathcal{K}))$,
(ii) $\beta_G \otimes_{C_0(G/\mathcal{K})} \alpha_G = \gamma_G = KK_G(\mathbb{C}, \mathbb{C})$, where γ_G is an element in $KK_G(\mathbb{C}, \mathbb{C})$.

(For a semisimple Lie group G or for $G = \mathbb{R}^n$, a construction of the Mishchenko element β_G can be found in [20].) We assume the following definition: a Lie group G is said to be K-amenable if $\gamma_G = 1$. The non-amenable groups $SO(n, 1)$ and $SU(n, 1)$ are K-amenable [21, 22].

Note the result of the calculation given in [23–25]: $K_*(C^*(\mathbb{Z}, \sigma)) \cong K_*(C^*(\mathbb{Z}^n)) \cong K_*(\mathbb{T}^n)$, which holds for any group two-cocycle σ on $\mathbb{Z}^n$. This calculation leads to the twisted group C^*-algebras $C^*(\mathbb{Z}, \sigma)$ (noncommutative tori). Such generalization has been given for K-groups of the twisted group C^*-algebras of uniform lattice in solvable groups [26]. Let Γ be a uniform lattice in a solvable Lie group G. The following result generalized the result of [26]. Let Γ be a lattice in a K-amenable Lie group G; then [20]

$$K_*(C^*(\Gamma, \sigma)) \cong K^{*+\dim(C/\mathcal{K})}(\Gamma\backslash G/\mathcal{K}, \delta(B_\sigma)), \tag{14}$$

where $K^{*+\dim G}(\Gamma\backslash G, \delta(B_\sigma))$ denotes the twisted K-theory (see [27]) of a continuous trace C^*-algebra B_σ with spectrum $\Gamma\backslash G, \sigma$ is any multiplier on Γ, while $\delta(B_\sigma) \in H^3(\Gamma\backslash G, \mathbb{Z})$ denotes the Dixmier–Douady invariant of B_σ [28]. Let Γ be a lattice in a K-amenable Lie group G. Then the following formulae hold [20]

$$K_*(C^*(\Gamma, \sigma)) \cong K^{*+\dim G}(C_r^*(\Gamma, \sigma)), \tag{15}$$

$$K_*(C^*(\Gamma, \sigma)) \cong K^{*+\dim G/\mathcal{K}}(\Gamma\backslash G/\mathcal{K}, \delta(B_\sigma)). \tag{16}$$

These formulae have been obtained with the help of K-amenability results [21] and the stabilization theorem [26]. When $\Gamma = \Gamma_g$ is a fundamental group of a Riemannian surface $X_\Gamma = \Sigma_g$ of genus $g > 0$, the Dixmier–Douady class $\delta(B_\sigma)$ is trivial and we get

$$K_0(C^*(\Gamma_g, \sigma)) \cong K^0(\Sigma_g) \cong \mathbb{Z}^2 \qquad K_1(C^*(\Gamma_g, \sigma)) \cong K^1(\Sigma_g) \cong \mathbb{Z}^{2g}, \tag{17}$$

which hold for any multiplier σ on Γ_σ.

Just some basics on the lower algebraic K-groups. Let X_Γ be a real compact oriented three-dimensional hyperbolic manifold. Its fundamental group Γ comes with maps to $PSL(2, \mathbb{C}) \equiv SL(2, \mathbb{C})/\{\pm Id\}$; therefore, in general, one gets a class in $H_3(GL(\mathbb{C}))$. The following result holds [29–31] for a field $\mathcal{F}$:

$$K_j(\mathcal{F}) \cong H_j(GL(j, \mathcal{F}))/H_j(GL(j-1, \mathcal{F}). \tag{18}$$

The group $K_3(\mathcal{F})$ is built out of $K_3(\mathcal{F})$ and the Bloch group $\mathfrak{B}(\mathcal{F})$. Since we are looking for $H_3(GL(2, \bullet))$, the homology invariant of a hyperbolic 3-manifold should live in the Bloch group $\mathfrak{B}(\bullet)$. The following result confirmed that statement [32]: a real oriented finite-volume hyperbolic 3-manifold $X = X_\Gamma$ has an invariant $\beta(X) \in \mathfrak{B}(\mathbb{C})$. Actually $\beta(X) \in \mathfrak{B}(\mathcal{F})$ for an associated number field $\mathcal{F}(X)$. In fact, under the (normalized) Bloch regulator $\mathfrak{B}(\mathbb{C}) \to \mathbb{C}/\mathbb{Q}$, the invariant $\beta(X)$ goes to $\{(2/\pi)\,\mathrm{vol}(X_\Gamma) + 4\pi\sqrt{-1}CS(X_\Gamma)\}$. Let us assume that $\mathcal{F}(X_\Gamma)$ can be embedded in $\mathbb{C}$ as an imaginary quadratic extension of a totally real number field; then $CS(X_\Gamma)$ is rational (conjecturably, $CS(X)$ is irrational if $\mathcal{F}(X)\cap\overline{\mathcal{F}(X)} \subset \mathbb{R}$ [33]). The volume and the Chern–Simons invariants can be combined into a single complex invariant [34]. Taking into account the Thurston classification of all three possible geometries, $\Gamma_n\backslash G_n$, this invariant can be presented in the form $\exp\left\{\bigcup_{\ell=1}^{\infty}[(2/\pi)\,\mathrm{Vol}(\Gamma_{n\ell}\backslash G_{n\ell}) + 4\pi\sqrt{-1}CS(\Gamma_{n\ell}\backslash G_{n\ell})]\right\}$.

To finish, if we adhere to the intuitive requirement that only irreducible manifolds have to be taken into account (supersymmetry surviving arguments being in favour of this requirement [10]), then the manifolds modelled on $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$ have to be excluded from Thurston's list. There is only a finite number of manifolds of the form $\Gamma\backslash\mathbb{R}^N$, $\Gamma\backslash\mathbb{S}^N$, for any N [35]. It seems that in QFT the most important contribution to the vacuum persistence amplitude should be given by the hyperbolic geometry, the other geometries appearing only for a small number of exceptions [36]. Indeed, many 3-manifolds are hyperbolic (according to a famous theorem by Thurston [34]). For example, the complement of a knot in $\mathbb{S}^3$ admits a hyperbolic structure unless it is a torus or a satellite knot. Moreover, from Mostow's rigidity theorem [37], any geometric invariant of a hyperbolic 3-manifold is a topological invariant.

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